Stokes Problem with Stick-Slip Boundary Conditions

Application of the SCD Semismooth* Newton Method

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We present the 3D Stokes problem with Navier-Tresca boundary conditions, formulated as a variational inequality with an equality constraint. The problem is approximated using the mixed FEM, yielding a generalized equation. Then we implement a variant of the SCD semismooth* Newton method. Numerical experiments demonstrate the efficiency of this approach.

Stokes problem

$$\begin{aligned} -2\nu\nabla\cdot\nabla_S\,\boldsymbol{u} + \nabla p &= \mathtt{f} && \text{in } \Omega\\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega\\ \boldsymbol{u} &= \boldsymbol{u}_D && \text{on } \gamma_D\\ u_n &= 0 && \text{on } \gamma_S\\ \|\sigma_t + \kappa\boldsymbol{u}_t\| &\le g && \text{on } \gamma_S\\ \sigma_t\cdot\boldsymbol{u}_t + g\|\boldsymbol{u}_t\| + \kappa\boldsymbol{u}_t\cdot\boldsymbol{u}_t &= 0 && \text{on } \gamma_S \end{aligned}$$

where

- $\Omega\subset\mathbb{R}^d,\quad \partial\Omega = \overline{\gamma}_D\cup\overline{\gamma}_N\cup\overline{\gamma}_S$
- $\nabla_S\,\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top)\in\mathbb{R}^d\times\mathbb{R}^d$
- $\sigma = (2\nu\,\nabla_S\,\boldsymbol{u} - p\mathtt{I})\mathtt{n}\quad$ on $\partial\Omega$.
- Slip bound $g\ge 0$, Adhesion $\kappa > 0$
- Approximation by FEM P1-bubble/P1 pair
- Bubble components are eliminated

Generalized equation

Using the definition of the subdifferential of the function q, we obtain the following generalized equation:

$$0\in F(u,p) = \begin{pmatrix} A_\kappa & B^\top\\ B & -E\end{pmatrix}\begin{pmatrix} u\\ p\end{pmatrix} - \begin{pmatrix} b\\ c\end{pmatrix} + \partial q(u,p), \tag{1}$$

with

$$q(u,p) = \sum_{i\in\mathcal{N}}\left(g_i\|T^iu^i\| + \delta_{S^i}(u^i)\right),\quad S^i := \left\{v\in\mathbb{R}^3 : N^iv = 0\right\},$$

where $\mathcal{N} := \{1,\dots,n_s\}$ is the index set of the nodes lying on γ_S and the indicator function δ_S is given by

$$\delta_S(x) = \begin{cases} 0 & \text{if } x\in S,\\ \infty & \text{if } x\notin S,\end{cases}$$

and ensures thus the impermeability condition.

The generalized equation (1) can be simplified in the form

$$0\in F(x) = H(x) + \partial q(x), \tag{2}$$

where $\partial q(x)$ is a multifunction.

SCD semismooth* Newton method

The SCD semismooth Newton* method, which we use to solve (1), is based on a property of the multifunction $G:\mathbb{R}^m\rightrightarrows\mathbb{R}^m$ that we call SCD semismooth*.

Definition 1. *We say that $G:\mathbb{R}^m\rightrightarrows\mathbb{R}^m$ is* SCD semismooth* at $(\bar{x},\bar{y})\in\operatorname{gph} G$ *if G has the SCD property around $(\bar{x},\bar{y})$ and for every $\epsilon > 0$ there is some $\delta > 0$ such that*

$$|\langle x^*, x-\bar{x}\rangle - \langle y^*, y-\bar{y}\rangle| \le \epsilon\|(x,y)-(\bar{x},\bar{y})\|\|(x^*,y^*)\| \tag{3}$$

holds for all $(x,y)\in\operatorname{gph} G\cap\mathcal{B}_\delta(\bar{x},\bar{y})$ and all $(y^*,x^*)\in L^*$, $L^*\in\mathcal{S}^*G(x,y)$.

To define the SCD semismooth*, it is also necessary to introduce the concepts of the subspace-containing (SC) derivative $\mathcal{S}G$ and the adjoint SC derivative $\mathcal{S}^*G$.

Definition 2. *Let $G:\mathbb{R}^m\rightrightarrows\mathbb{R}^m$ be a set-valued mapping. Then:*

1. *The* subspace containing (SC) derivative $\mathcal{S}G:\operatorname{gph} G\rightrightarrows\mathcal{Z}_m$ *of G is defined by*

$$\mathcal{S}G(x,y) := \{L\in\mathcal{Z}_m \mid \exists(x_k,y_k)\xrightarrow{\mathcal{O}_G}(x,y) : \lim_{k\to\infty} d_{\mathcal{Z}}\big(L, T_{\operatorname{gph} G}(x_k,y_k)\big) = 0\}.$$

2. *The* adjoint SC derivative $\mathcal{S}^*G:\operatorname{gph} G\rightrightarrows\mathcal{Z}_m$ *of G is given by*

$$\mathcal{S}^*G(x,y) = \{L^* \mid L\in\mathcal{S}G(x,y)\}.$$

Acknowledgements

This research has been conducted with the financial support of the European Union under the REFRESH – Research Excellence for Region Sustainability and High-tech Industries project number CZ.10.03.01/00/22_003/0000048 through the Operational Programme Just Transition.

SCD semismooth Newton* method for Stokes problem

1. Choose a starting point $\boldsymbol{x}^0 = (\boldsymbol{u}^k,\mathtt{p}^k)$, set the iteration counter $k := 0$
2. If $\mathtt{0}\in F\left(\boldsymbol{x}^0\right)$, stop the algorithm.
3. **Approximation step:** Compute
   - Compute Moreau-Yoshida problem to find pair $(\boldsymbol{z},\boldsymbol{z}^*)\in\operatorname{gph}\partial q$ by
   $$z^{(k)} := \operatorname{prox}_{\lambda q}(x^{(k)} - \lambda H(x^{(k)})),\quad z^{*(k)} := \frac{x^{(k)} - z^{(k)}}{\lambda} - H(x^{(k)}),$$
   where $\operatorname{prox}_\varphi(x) := \arg\min_z\left\{\frac{1}{2}\|z-x\|^2 + \varphi(z)\right\}$ and $\lambda > 0$ is fixed
   - Create matrices $\mathtt{P}$, $\mathtt{W}$
   $$rge(\mathtt{P},\mathtt{W})\in\mathcal{S}(\partial q)(\boldsymbol{z},\boldsymbol{z}^*) = \mathcal{S}^*(\partial q)(\boldsymbol{z},\boldsymbol{z}^*),$$
4. **Newton step:** Compute the Newton direction $(\delta\boldsymbol{u},\delta\mathtt{p})$ as solution of the linear system as Schur complement system of the system:
   $$\left(\mathtt{P}\begin{pmatrix} A_\kappa & B^\top\\ B & -\mathtt{E}\end{pmatrix} + \mathtt{W}\right)\begin{pmatrix}\delta\boldsymbol{u}\\ \delta\mathtt{p}\end{pmatrix} = (\gamma\mathtt{P} + \mathtt{W})\begin{pmatrix}\Delta\boldsymbol{u}\\ \Delta\mathtt{p}\end{pmatrix}$$
   Compute the new iterate via $\boldsymbol{u}^{k+1} = \boldsymbol{u}^k + \delta\boldsymbol{u}^k$ and $\mathtt{p}^{k+1} = \mathtt{p}^k + \delta\mathtt{p}^k$
5. Set $k := k+1$ and go to 2.

Numerical Examples

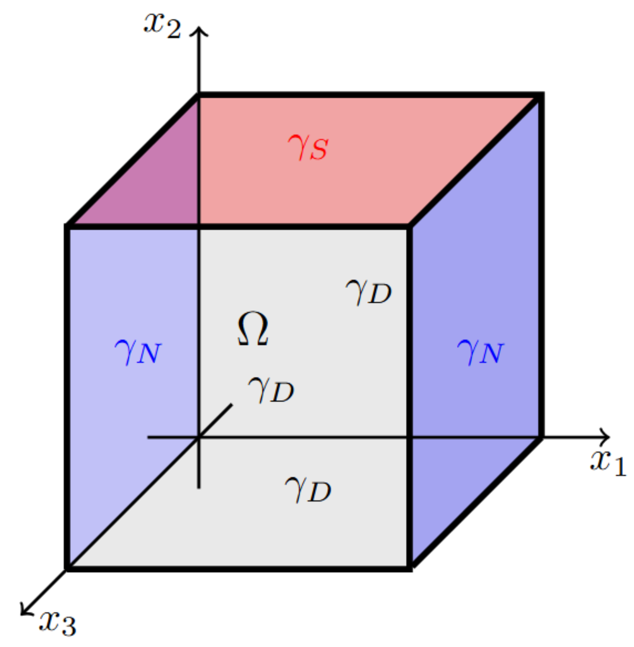
- $\Omega = (0,1)^3$
- $\partial\Omega = \overline{\gamma}_D\cup\overline{\gamma}_N\cup\overline{\gamma}_S$
- $\nu = 0.9,\ \kappa = 5$
- $\mathtt{f} = -2\nu\,\nabla\cdot\nabla_S\,\boldsymbol{u}_{exp} + \nabla p_{exp}$ with

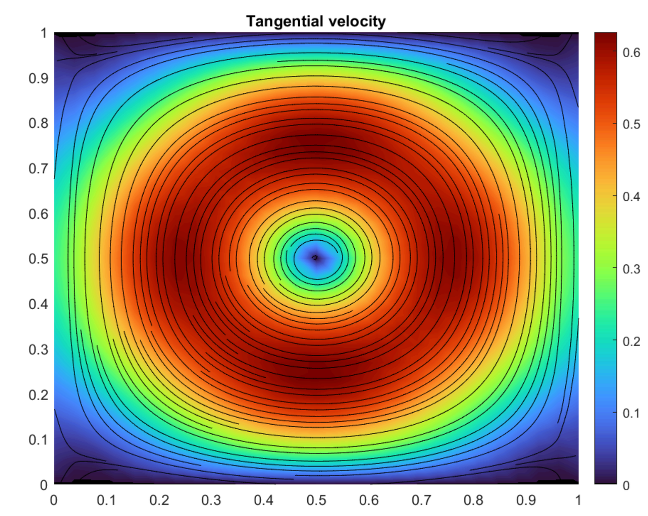
$u_1 = 4(1-\cos(2\pi x))\sin(2\pi y)z(1-z)$

$u_2 = 4\sin(2\pi x)(\cos(2\pi y) - 1)z(1-z)$

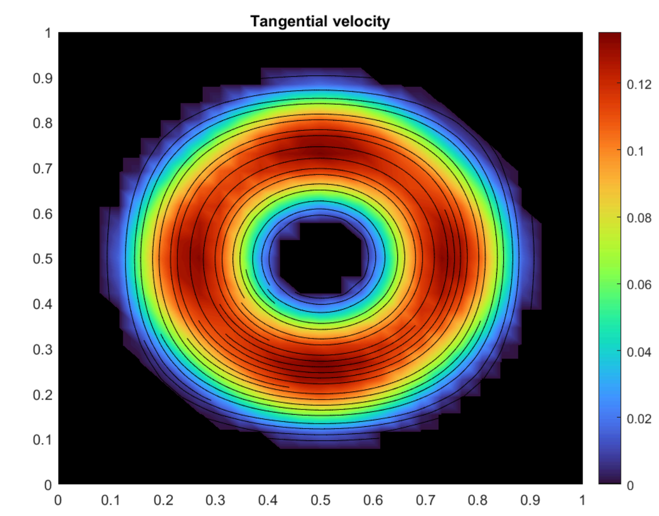
$u_3 = 0$

$p = 2\pi\left\{-\cos(2\pi x) + 2\cos(2\pi y) - \cos(2\pi z)\right\}$





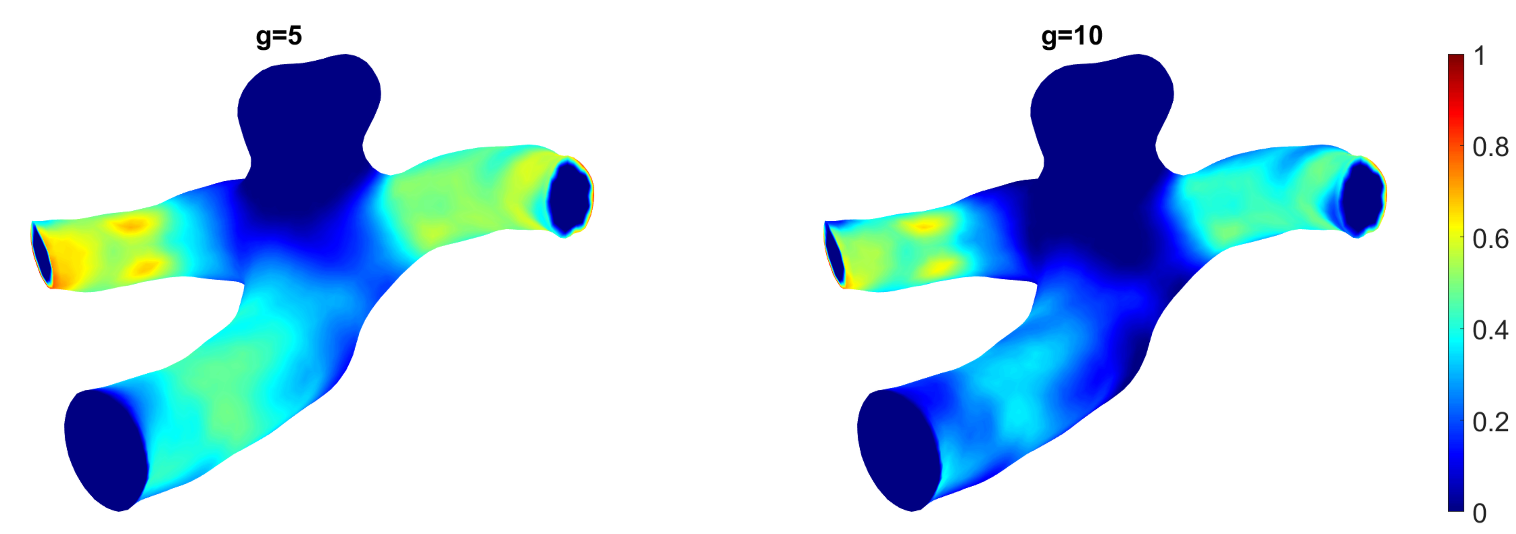
Tangential velocity for $\boldsymbol{g} = 0$



Tangential velocity for $\boldsymbol{g} = 5$

$\kappa = 5$	$g = 0$			$g = 5$			$g = 10$		
$n_p/n_t/n_s$	It	It_{in}	CPU	It	It_{in}	CPU	It	It_{in}	CPU
3375/ 13720 / 195	5	104	0.89	7	168	1.17	4	94	0.70
6859/ 29160 / 323	5	115	3.36	6	136	3.58	4	92	2.45
12167/ 53240 / 483	5	133	7.38	8	265	12.88	4	97	5.45
15625/ 69120 / 575	5	141	11.36	7	255	17.30	4	99	8.97
19683/ 87880 / 675	5	148	18.44	8	263	25.72	4	102	18.69

Cerebral Aneurysm from the Vascular Model Repository [www.vascularmodel.com]



Tangent velocity

References

[1] H. Gfrerer V. Arzt P. Beremlijski and J.V. Outrata. "On the application of the SCD semismooth* Newton method to solving Stokes problem with stick-slip boundary conditions". In: *Set-Valued and Variational Analysis (submitted)* (2026).